PREPROJECTIVE ALGEBRAS AND LUSZTIG'S NILPOTENT VARIETY

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Abstract. These are notes for a talk I gave at the BIREP summer school on preprojective algebras, held August 26-30 2013 in Bad Driburg. Most of the material is not original, though some of the proofs have been simplified. In particular Section 6.2 contains a new proof of Lusztig's nilpotent variety being isotropic.

1. Introduction

In this talk we will start the geometric study of representations of a fixed quiver Q. For every dimension vector α we will construct an affine variety $\mathrm{Rep}(Q, \alpha)$ acted on by a linear algebraic group $GL(\alpha)$, such that the orbits classify representations of the quiver of dimension vector α. This will allow us to use methods from algebraic geometry in studying the quiver, and one can even associate a Hamiltonian action on a symplectic manifold to this setting, such that the preprojective algebra (and its deformations) show up naturally when studying certain fibers of the corresponding moment map. The variety associated to the preprojective algebra can be decomposed into irreducible components and if we restrict to finite type quivers, these irreducible components turn out to be in bijection with the orbits of $GL(\alpha)$, and thus with the isomorphism classes of representations of Q. These in turn correspond to a basis of the generic (twisted) Ringel-Hall algebra associated to Q, which is isomorphic to the positive part of the quantum group of the underlying graph of Q. Through a special base change, Lusztig was able to construct a canonical basis for this quantum group which has many interesting properties. This basis was constructed using perverse sheaves on the representation variety, and these natural bijections thus establish a link between the algebraic geometry of the variety, and the symplectic geometry of the cotangent bundle, a theme cropping up in many places (e.g. homological mirror symmetry). If Q is now any quiver without loops, then the full representation variety of the preprojective algebra turns out to be badly behaved, and one needs to pass to Lusztig's nilpotent variety. This variety is a Lagrangian subvariety in the cotangent bundle, and all of the natural bijections above have analogues in this more general setting, by constructing from the decomposition into irreducible components a $\mathfrak{g}$-crystal, which can be shown to come from a crystal basis, again obtaining a bijection with the canonical basis.

Date: August 26-30, 2013.

2. A (very) brief account of symplectic geometry

Throughout most of the talk, k will be an algebraically closed field of characteristic 0, which one can safely assume to be $\mathbb{C}$. We will very briefly review a couple of notions we need from symplectic geometry. A symplectic variety is a variety X equipped with a non-degenerate, closed 2-form ω. These varieties are always even-dimensional and orientable, and they all look locally the same (by the famous Darboux theorem). Some of the most important examples of symplectic varieties are cotangent bundles T^*X. If π denotes the bundle map

$$\pi : T^*X \to X : p = (x, \xi) \mapsto x$$

then one defines the Liouville (or tautological) 1-form on T^*X by

$$\alpha_p = (d\pi_p)^*\xi \in T_p^*(T^*X).$$

The canonical symplectic form ω on T^*X is then defined to be

$$\omega = -d\alpha,$$

which locally looks like

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i.$$

The most important class of submanifolds of symplectic manifolds are the Lagrangian submanifolds (e.g. Fukaya categories, graphs of symplectomorphisms, fibers of integrals of motion, WKB expansion). If (X, ω) is a symplectic variety, then a subvariety Y is Lagrangian if $\omega|_Y = 0$, and $\dim Y = \frac{1}{2}\dim X$ (these equalities are taken to hold at all smooth points). The last notion we need is that of a Hamiltonian G-space. Suppose we start with a symplectic variety (X, ω) equipped with a symplectic action of a connected linear algebraic group G, which just means that ω is G-invariant. Then this action is said to be Hamiltonian if there exists a map

$$\mu : X \to \mathfrak{g}^*,$$

such that

- For any $\theta \in \mathfrak{g}$, let $\theta^\#$ denote the fundamental vector field associated to the action, i.e. $\theta^\#_x = (dR_x)_e(\theta)$, where $R_x : G \to X : g \mapsto g \cdot x$. Then the equation
$$d\mu^\theta = i_{\theta^\#}\omega$$
has to be satisfied, where $\mu^\theta : M \to \mathbb{R} : p \mapsto \langle \mu(p), \theta\rangle$, and i denotes the interior product. This says that μ^θ is a Hamiltonian function for the vector field $\theta^\#$.
- μ is equivariant with respect to the given action of G on X and the coadjoint action of G on $\mathfrak{g}^*$ (i.e. $\mathrm{Ad}^*_g(f)(\theta) = f(\mathrm{Ad}(g)^{-1}\theta)$):
$$\mu \circ g = \mathrm{Ad}^*_g \circ \mu$$

In this case, the quadruple (X, ω, G, μ) is called a Hamiltonian G-space, and μ is called the moment map (used for symplectic reductions, symplectic cuts, sums, abstract versions of Noether's theorem,...).

3. Geometry of quiver representations

This section is based on Crawley-Boevey [1]. Assume $Q = (Q_0, Q_1, h(ead), t(ail))$ is a quiver (possibly with loops and oriented cycles). Arrows will always be denoted $a \in Q$. For $\alpha \in \mathbb{N}^{Q_0}$, define

$$\mathrm{Rep}(Q,\alpha) = \prod_{a:i\to j} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha_j}) = \prod_{a:i\to j} \mathrm{Mat}_{\alpha_j\times\alpha_i}(k),$$

the representation space of Q of dimension vector α, and denote its elements by $x = (x_a)_{a\in Q}$. Since we are dealing with quivers without relations at the moment, this is just $\mathbb{A}^{\sum_{a:i\to j}\alpha_i\alpha_j}$ as algebraic variety. Define the irreducible (thus connected) linear algebraic group

$$GL(\alpha) = \prod_{l\in Q_0} GL(\alpha_l),$$

which acts linearly on each component $\mathrm{Mat}_{\alpha_j\times\alpha_i}(k)$ of $\mathrm{Rep}(Q,\alpha)$ via conjugation:

$$(g_l)_l \cdot x_a = g_j x_a g_i^{-1}$$

Since scalar matrices act trivially, this action factors through the action of the quotient group

$$PGL(\alpha) = GL(\alpha)/k^*\mathrm{Id}_\alpha.$$

Notice there is only one copy of k^*! It is then clear that there is a bijection between the isomorphism classes of quiver representations of dimension vector α, and the orbits of $PGL(\alpha)$ in $\mathrm{Rep}(Q,\alpha)$. It is tempting to use the notation

$$\mathrm{Rep}(Q,\alpha)/PGL(\alpha),$$

though one should be aware of the fact that without some sort of stability conditions, the orbit space can not be made into a variety! It can however be given sense as an algebraic stack, though we will not pursue this avenue. Denote by Tr the non-degenerate trace pairing

$$\mathrm{Tr} : \mathrm{Mat}_{m\times n} \times \mathrm{Mat}_{n\times m} \to k : (M,N) \mapsto \mathrm{trace}(MN).$$

We are interested in $T^*\mathrm{Rep}(Q,\alpha)$, the cotangent bundle to the representation variety. Since $\mathrm{Rep}(Q,\alpha)$ is just an affine space, elements of this bundle consist of tuples $(x, (f_a : \mathrm{Mat}_{\alpha_j\times\alpha_i}(k) \to k)_{a\in Q})$. Using the trace pairing, these correspond to tuples $(x, (x_{a^*})_{a\in Q})$, where x_{a^*} is an $\alpha_i \times \alpha_j$-matrix if $a : i \to j$. Denoting by $\overline{Q}$ the quiver obtained from Q by adding an arrow $a^* : j \to i$ for every arrow $a : i \to j$, we see that

$$T^*\mathrm{Rep}(Q,\alpha) \cong \mathrm{Rep}(\overline{Q},\alpha).$$

As we've seen, the cotangent bundle to a variety always carries a symplectic structure, coming from the Liouville 1-form. If $\xi = (x_{a^*})_{a\in Q}$, then the bundle map is just

$$\pi : \mathrm{Rep}(\overline{Q},\alpha) \to \mathrm{Rep}(Q,\alpha) : p = (x,\xi) \mapsto x,$$

so the Liouville form becomes

$$\alpha_p(y) = \xi(d\pi_p(y)) = \sum_{a\in Q} \mathrm{Tr}(x_{a^*} y_a),$$

for $y \in T_p(T^*\mathrm{Rep}(Q,\alpha)) \cong T^*(\mathrm{Rep}(Q,\alpha))$. Using the invariant formula for the derivative of a one form:

$$d\alpha(Y,Z) = Y(\alpha(Z)) - Z(\alpha(Y)) - \alpha([Y,Z]),$$

the canonical symplectic form becomes

$$\omega(y,z) = -d\alpha(y,z) = -\sum_{a\in Q}\mathrm{Tr}(y_{a^*}z_a) + \sum_{a\in Q}\mathrm{Tr}(z_{a^*}y_a),$$

where y, z are identified with their respective constant vector fields. Notice that this form is invariant for the induced action of $PGL(\alpha)$ on $\mathrm{Rep}(\overline{Q},\alpha)$, so the formula makes sense and the action is symplectic. We claim that this action is Hamiltonian, turning $\mathrm{Rep}(\overline{Q},\alpha)$ into a Hamiltonian $PGL(\alpha)$-space. The Lie algebra of $PGL(\alpha) = PSL(\alpha)$ is equal to

$$\mathfrak{sl}(\alpha) = \mathrm{End}(\alpha)_0 = \{(\theta_i)_i \in \prod_{i\in Q_0}\mathrm{Mat}_{\alpha_i\times\alpha_i} \mid \sum_{i\in Q_0}\mathrm{Tr}(\theta_i) = 0\}.$$

Notice that it is not each matrix separately that is traceless, since there is only one copy of k^* acting trivially. Using the trace pairing, $\mathfrak{sl}(\alpha)$ and $\mathfrak{sl}(\alpha)^*$ can be identified, and we claim that the map

$$\mu_\alpha : \mathrm{Rep}(\overline{Q},\alpha) \to \mathrm{End}(\alpha)_0,$$

defined by

$$\mu_\alpha(x)_i = \sum_{\substack{a\in Q\\ h(a)=i}} x_a x_{a^*} - \sum_{\substack{a\in Q\\ t(a)=i}} x_{a^*}x_a$$

is a moment map. Since for matrices, the coadjoint action is just conjugation, it is obvious that μ is equivariant. To check that it is globally Hamiltonian, remember we need to verify that

$$d\mu^\theta = i_{\theta^\#}\omega, \tag{3.1}$$

where

$$\mu^\theta : \mathrm{Rep}(\overline{Q},\alpha) \to k : x \mapsto \langle \mu(x), \theta\rangle.$$

In our case, it is thus given by

$$\mu^\theta(x) = \sum_{i\in Q_0}\mathrm{Tr}(\theta_i\mu_\alpha(x)_i).$$

Since the action is by conjugation, the fundamental vector fields are given by the Lie bracket:

$$(\theta^\#_x)_a = [\theta, x]_a = \theta_{h(a)}x_a - x_a\theta_{t(a)}, a \in \overline{Q}.$$

To check (3.1), we first rewrite μ^θ

$$\begin{aligned}\mu^\theta(x) &= \sum_{i\in Q_0}\sum_{\substack{a\in Q\\ h(a)=i}}\mathrm{Tr}(\theta_i x_a x_{a^*}) - \sum_{i\in Q_0}\sum_{\substack{a\in Q\\ t(a)=i}}\mathrm{Tr}(\theta_i x_{a^*}x_a)\\ &= \sum_{a\in Q}\mathrm{Tr}(\theta_{h(a)}x_a x_{a^*}) - \mathrm{Tr}(\theta_{t(a)}x_{a^*}x_a)\end{aligned}$$

Now since the trace is linear and θ is constant, all one needs to compute the derivative of μ^θ is the formula

$$dm_{X,Y}(A,B) = XB + AY,$$

where m denotes matrix multiplication. In conclusion, the LHS becomes

$$d\mu^\theta_x(y) = \sum_{a\in Q}\mathrm{Tr}(\theta_{h(a)}(x_a y_{a^*} + y_a x_{a^*})) - \mathrm{Tr}(\theta_{t(a)}(y_{a^*}x_a + x_{a^*}y_a)),$$

whereas for the RHS, we have:

$$\begin{aligned} i_{\theta^\#}(\omega)_x(y) &= \omega([\theta, x], y) \\ &= \sum_{a\in Q} \mathrm{Tr}([\theta, x]_a y_{a^*}) - \mathrm{Tr}([\theta, x]_{a^*} y_a) \\ &= \sum_{a\in Q} \mathrm{Tr}(\theta_{h(a)} x_a y_{a^*} - x_a \theta_{t(a)} y_{a^*}) - \mathrm{Tr}(\theta_{h(a^*)} x_{a^*} y_a - x_{a^*} \theta_{t(a^*)} y_a) \\ &= \sum_{a\in Q} \mathrm{Tr}(\theta_{h(a)} x_a y_{a^*} - \theta_{t(a)} y_{a^*} x_a) - \mathrm{Tr}(\theta_{t(a)} x_{a^*} y_a - \theta_{h(a)} y_a x_{a^*}), \end{aligned}$$

proving the claim.

4. (Deformed) preprojective algebras

Looking back at $\mathfrak{sl}(\alpha)$, it is obvious that the invariants under the $PGL(\alpha)$-action are the $\theta = (\theta_i)_i$, such that each θ_i is a scalar matrix. These will be identified with the $\lambda \in k^{Q_0}$, satisfying the equation $\lambda \cdot \alpha = \sum_{i\in Q_0} \lambda_i \alpha_i = 0$. It turns out that the fibers of the moment map over these λ lead to interesting geometry. References for this part include Crawley-Boevey(+ Holland) [1, 2]. We start off by introducing the preprojective algebra and its deformations. For λ as before, define

$$\Pi^\lambda Q = k\overline{Q}/(\sum_{a\in Q}[a, a^*] - \sum_{i\in Q_0} \lambda_i e_i),$$

where e_i denotes the trivial path at the vertex i, and $k\overline{Q}$ the path algebra of the double quiver. The algebra $\Pi^\lambda Q$ is known as a deformed preprojective algebra of weight λ, and the special case $\lambda = 0$ is the original preprojective algebra, which we will denote ΠQ. The ideal of relations can equivalently be described as the ideal having a generator

$$\sum_{\substack{a\in Q\\ h(a)=i}} aa^* - \sum_{\substack{a\in Q\\ t(a)=i}} a^*a - \lambda_i e_i$$

for every vertex i, making it clear that there is an isomorphism between the scheme theoretic fiber of the moment map, and the representation space $\mathrm{Rep}(\Pi^\lambda Q, \alpha)$. To get at nice results, we will start investigating the image of $\mu_\alpha^{-1}(\lambda)$ under the bundle map

$$\pi : \mu_\alpha^{-1}(\lambda) \to \mathrm{Rep}(Q, \alpha)$$

Our first observation about the fibers of the moment map immediately implies that if $\lambda \cdot \alpha \neq 0$, there exist no representations of $\Pi^\lambda Q$ of dimension vector α. We will now derive a partial converse to this result. First remember that for $x \in \mathrm{Rep}(Q, \alpha)$, there is a four-term exact sequence

$$0 \to \mathrm{End}_Q(x) \to \mathrm{End}(\alpha) \xrightarrow{f} \mathrm{Rep}(Q, \alpha) \to \mathrm{Ext}^1_Q(x, x) \to 0,$$

where $\mathrm{End}(\alpha) = \{(\theta_i)_{i\in Q_0} \mid \theta_i \in \mathrm{Mat}_{\alpha_i\times\alpha_i}(k)\}$, and

$$f((\theta_i)_i)_a = \theta_{h(a)} x_a - x_a \theta_{t(a)}.$$

The kernel of this map is obviously $\mathrm{End}_Q(x)$, and Ringel's formula for the Tits form gives the right dimension for the cokernel. The natural isomorphism with the

Ext-space comes from the standard projective resolution of a kQ-module, see [2]. By dualizing and using the trace pairing again, we get the exact sequence

$$0 \to \mathrm{Ext}^1_Q(x,x)^* \to \mathrm{Rep}(Q^o,\alpha) \xrightarrow{c} \mathrm{End}(\alpha) \xrightarrow{t} \mathrm{End}_Q(x)^* \to 0,$$

where Q^o denotes the opposite quiver, which is just Q, with orientations of the arrows reversed, and

$$c((y_{a^*})_{a^*}) = \sum_{a\in Q} [x_a, y_{a^*}]$$
$$[t((\theta_i)_i)]((f_i)_i) = \sum_i \mathrm{Tr}(\theta_i f_i)$$

This allows for a proof of

Theorem 4.1. *For a weight $\lambda \in k^{Q_0}$, and a representation $x \in \mathrm{Rep}(Q,\alpha)$ the following are equivalent*

(1) *x lifts to a representation of $\Pi^\lambda Q$ of dimension vector α*
(2) *For all dimension vectors β of direct summands x' of x we have $\lambda\cdot\beta = 0$*

In this case, $\pi^{-1}(x) \cong \mathrm{Ext}^1_Q(x,x)^*$.

Proof. If x lifts to a representation of $\Pi^\lambda Q$, then $\lambda \in \mathrm{Im}(c) = \mathrm{Ker}(t)$, so

$$\sum_i \lambda_i \mathrm{Tr}(f_i) = 0,$$

for any $f = (f_i)_i \in \mathrm{End}_Q(x)$. Taking x' to be a direct summand of x, let f be the map

$$x \twoheadrightarrow x' \hookrightarrow x,$$

then obviously $\sum_i \lambda_i \mathrm{Tr}(f_i) = \sum_i \lambda_i\beta_i = 0$.

To prove the converse, we can suppose x' is an indecomposable representation of dimension vector β satisfying $\lambda\cdot\beta = 0$. Endomorphism rings of indecomposable representations are local algebras, so any endomorphism $(f_i)_i$ of x' is the sum of a scalar matrix and a nilpotent matrix (which has zero trace), so $\sum_i \lambda_i \mathrm{Tr}(f_i) = 0$. This means that $\lambda \in \mathrm{Ker}(t) = \mathrm{Im}(c)$, so x lifts. The last statement follows from exactness again, since $\mathrm{Ker}(c) = \pi^{-1}(x)$. □

To say more about the image of π, we need the following

Definition 4.2. If G is an algebraic group acting on a variety Y and $X \hookrightarrow Y$ is a G-stable constructible subset, then X can be decomposed as $X = \cup_d X_{(d)}$, where each piece $X_{(d)}$ is the union of orbits of dimension d. The number of parameters of X is

$$p(X) = \max_d(\dim(X_{(d)}) - d),$$

where $\dim(X_{(d)})$ denotes the dimension of $\overline{X_{(d)}}$.

This will be used in the following proposition, the last part of which we will need later on.

Proposition 4.3. *Let U be a $GL(\alpha)$-stable constructible subset of* $\mathrm{Rep}(Q,\alpha)$ *contained in* $\mathrm{Im}(\pi)$*; then*

$$\dim\, \pi^{-1}(U) = p(U) + \sum_{a\in Q} \alpha_{t(a)}\alpha_{h(a)}.$$

If U is in fact a single orbit, then $\pi^{-1}(U)$ is irreducible of dimension $\sum_{a\in Q}\alpha_{t(a)}\alpha_{h(a)}$.

Proof. Given $x \in U_{(d)}$, then by Theorem 4.1, the fiber $\pi^{-1}(x) \cong \mathrm{Ext}^1_Q(x,x)^*$, and

$$\dim\, \pi^{-1}(x) = \sum_{a\in Q}\alpha_{h(a)}\alpha_{t(a)} - \sum_i \alpha_i^2 + \dim\, \mathrm{End}_Q(x)$$

Furthermore,

$$\dim\, \mathrm{End}_Q(x) = \dim\, GL(\alpha) - \dim\, \mathcal{O}(x) = \sum_i \alpha_i^2 - d,$$

so $\dim\, \pi^{-1}(U_{(d)}) = (\dim\, U_{(d)} - d) + \sum_{a\in Q}\alpha_{h(a)}\alpha_{t(a)}$. Letting d vary, we immediately obtain the desired result. If U is a single orbit, suppose that $\pi^{-1}(U)$ contains $Z_1 \sqcup Z_2$, with Z_i a $GL(\alpha)$-stable open subset (in an irreducible set every open would be dense). Since $\pi(Z_i) = U$, the sets $\pi^{-1}(x)\cap Z_i$ are non-empty, disjoint and open in the set $\pi^{-1}(x)$, which is defined by linear equations in $\mathrm{Rep}(\overline{Q},\alpha)$ and is thus irreducible: contradiction! □

5. Quivers of finite type

Now suppose Q is a quiver of finite type, such that by Gabriel's theorem, the underlying graph of Q is an ADE Dynkin diagram Δ, and the map taking a representation to its dimension vector induces a bijection between the isomorphism classes of indecomposable kQ-modules and the positive roots of Δ. In particular the (finite) number of isomorphism classes of modules of dimension vector α is independent of the field k. We are interested in the irreducible components of $\mathrm{Rep}(\Pi Q,\alpha)$. It is obvious that the restriction of the bundle map

$$\pi : \mathrm{Rep}(\Pi Q,\alpha) \to \mathrm{Rep}(Q,\alpha)$$

is surjective; it even has a section, given by sending x to $(x,0)$. We will now construct a bijection between the irreducible components of $\mathrm{Rep}(\Pi Q,\alpha)$ and the orbits of $GL(\alpha)$ acting on $\mathrm{Rep}(Q,\alpha)$. The map

$$\mathrm{Irr}(\mathrm{Rep}(\Pi Q,\alpha)) \to \mathrm{Orbits}(\mathrm{Rep}(Q,\alpha))$$

is given as follows: if $Z \in \mathrm{Irr}(\mathrm{Rep}(\Pi Q,\alpha))$, then its image is irreducible, hence also its closure $\overline{\pi(Z)}$ is irreducible. Moreover, π is $GL(\alpha)$-equivariant, so $\overline{\pi(Z)}$ is invariant. So

$$\overline{\pi(Z)} = \cup_{i=1}^k \mathcal{O}_i = \cup\overline{\mathcal{O}_i},$$

and by irreducibility, $\overline{\pi(Z)} = \overline{\mathcal{O}_i}$, so we can associate a unique orbit to Z. For the map

$$\mathrm{Orbits}(\mathrm{Rep}(Q,\alpha)) \to \mathrm{Irr}(\mathrm{Rep}(\Pi Q,\alpha)),$$

take an orbit $\mathcal{O}$, and associate to it $\overline{\pi^{-1}(\mathcal{O})}$. By the last proposition, $\overline{\pi^{-1}(\mathcal{O})}$ is irreducible and

$$\dim\, \pi^{-1}(\mathcal{O}) = \dim\, \overline{\pi^{-1}(\mathcal{O})} = \dim\, \mathrm{Rep}(Q,\alpha) = \frac{1}{2}\dim\, \mathrm{Rep}(\overline{Q},\alpha).$$

Using finiteness of the number of orbits again

$$\text{Rep}(\Pi Q), \alpha) = \bigcup_{\mathcal{O}} \pi^{-1}(\mathcal{O}) = \bigcup_{\mathcal{O}} \overline{\pi^{-1}(\mathcal{O})},$$

which is the decomposition into irreducible components. So we see that there is a bijection between the irreducible components of $\text{Rep}(\Pi Q, \alpha)$ and the orbits of $\text{Rep}(Q, \alpha)$, which correspond to the isomorphism classes of kQ-modules of dimension vector α.

5.1. **The link with quantum groups.** We will briefly sketch the connection with $\mathcal{U}_q(\mathfrak{n}_+(\Delta))$, a deformation of the universal enveloping algebra of the positive part of the associated Lie algebra, based on Ringel [6]. Let $(a_{ij})_{ij}$ be the (symmetric) Cartan matrix to Δ, i.e. the $n \times n$-matrix (n being the number of vertices) with entries $a_{ii} = 2$, and $a_{ij} = -1$ if there is an (unoriented) edge $i \to j$. By Serre's theorem, this matrix gives rise to a presentation of the corresponding simple Lie algebra with $3n$ generators h_i, e_i and f_i, corresponding to the Cartan subalgebra, the positive and negative roots, and thus also to a presentation of its universal enveloping algebra. Now let v be a formal variable, and set $q = v^2$. Then $\mathcal{U}'_q(\mathfrak{n}_+(\Delta))$ is defined to be the $\mathbb{Q}(v)$-algebra generated by $E_1, \ldots, E_n$, satisfying the relations

$$\begin{aligned} E_iE_j - E_jE_i &= 0 \quad \text{if } a_{ij} = 0 \\ E_i^2E_j - (v + v^{-1})E_iE_jE_i + E_jE_i^2 &= 0 \quad \text{if } a_{ij} = -1 \end{aligned} \tag{5.1}$$

Though these relations might look strange at first flance, setting $v = 1$, these are just Serre's relations for the positive part of the classic enveloping algebra. Just like in the classical case, where one has Kostant's integral form, $\mathcal{U}'_q(\mathfrak{n}_+)$ has an integral form, defined as follows. Denote

$$E_i^{(m)} = \frac{1}{[m]!}E_i^m,$$

where $[m] = \frac{v^m - v^{-m}}{v - v^{-1}}$. Then $\mathcal{U}_q(\mathfrak{n}_+(\Delta))$ is the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $\mathcal{U}'_q(\mathfrak{n}_+(\Delta))$ generated by the $E_i^{(m)}$, $m \geq 0$. These algebras are widely studied, and have well-known applications in the study of 3-manifolds, knots, and the monodromy of differential equations.

On a seemingly unconnected note, define $\mathcal{H}(Q)$ to be the free $\mathbb{Z}[q]$-module with basis the set of maps $\Phi^+ \to \mathbb{N}$, Φ^+ denoting the positive roots of Δ. On this module, one can define a multiplication

$$\alpha_1 \diamond \alpha_2 = \sum_{\beta} \phi^{\beta}_{\alpha_1,\alpha_2}(q) \cdot \beta,$$

where the $\phi^{\beta}_{\alpha_1,\alpha_2}(q)$ are the so called Hall polynomials in $\mathbb{Z}[q]$, see Ringel [5]. For a finite field k, this generic Hall algebra specializes to the classical Hall algebra associated to the category of representations of a quiver over a finite field. Also, $\mathcal{H}(Q)$ is $\mathbb{Z}^n$-graded by assigning to a map $\alpha : \Phi^+ \to \mathbb{N}$ the dimension vector $\dim \alpha = \sum_a \alpha(a)a$. Notice that by Gabriel's theorem, an element $\Phi^+ \to \mathbb{N}$ of degree α corresponds to an isomorphism class $[N]$ of a kQ-module of dimension α, so there is also a natural bijection between the irreducible components of $\text{Rep}(\Pi Q, \alpha)$ and a basis of the α-graded part of $\mathcal{H}(Q)$. Now consider the $\mathbb{Z}[v, v^{-1}]$-module

$$\mathcal{H}_*(Q) = \mathcal{H}(Q) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[v, v^{-1}],$$

with multiplication defined by

$$[N_1] * [N_2] = v^{\langle \dim N_1, \dim N_2 \rangle} [N_1] \diamond [N_2].$$

This is again a $\mathbb{Z}^n$-graded algebra called the twisted Hall algebra, and one can prove that

$$\mathcal{U}_q(\mathfrak{n}_+(\Delta)) \cong \mathcal{H}_*(Q),$$

where the isomorphism is given by $E_i \mapsto [S_i]$, S_i being the simple module corresponding to the vertex i. Conceptually, this tells us that the rather mysterious Jimbo-Drinfeld relations 5.1 are the universal relations for comparing the numbers of composition series of modules over algebras with a prescribed quiver (this is how the multiplication in the Hall algebra is defined). In the other direction, $\mathcal{H}_*(Q)$ has a prescribed basis, so we get a basis and thus normal forms for elements of $\mathcal{U}_q(\mathfrak{n}_+(\Delta))$. This basis does however depend on the orientation of Q, and Lusztig proposed a base change leading to a basis having many nice properties (including orientation independence), called the canonical basis. This basis provides a uniform description, not only of the finite dimensional irreducible modules of $\mathfrak{g}$, but also gives a uniform choice of bases in all of them, and in their tensor products. The same basis was constructed more algebraically by Kashiwara, dubbing it the crystal basis of $\mathcal{U}_q(\mathfrak{n}_+(\Delta))$. Combining this with our previous bijections, we get a bijection

$$\mathrm{Irr}(\mathrm{Rep}(\Pi Q, \alpha)) \leftrightarrow \mathrm{Can}(\mathcal{U}_q(\mathfrak{n}_+(\Delta)))_\alpha \tag{5.2}$$

Interestingly, Lusztig constructed the canonical basis in a geometric way, and his proof was an early instance of categorification. The elements of the classical Hall algebra can be viewed as (constructible) functions on $\mathrm{Rep}(Q, \alpha)$, in such a way that the multiplication corresponds to a convolution product. Lusztig 'categorified' this by defining a category of constructible complexes of sheaves on the representation stack $\mathrm{Rep}(Q, \alpha)/PGL(\alpha)$ generated by the simple perverse sheaves. Using Grothendieck's Faisceaux-Fonctions correspondence, one can associate to each such perverse sheaf a constructible function on $\mathrm{Rep}(Q, \alpha)$, obtaining a basis of the classical Hall algebra, and thus of the quantum group. This is exactly the canonical basis, and so we see that this bijection establishes an interaction between the constructible sheaves on the variety, and Lagrangian subvarieties (see later) of its cotangent bundle.

6. The general case

From now on Q can again be any quiver (without loops). It should be clear that in the last part, we made use of the finite number of orbits in an essential way. We would like to have an analog of the bijection 5.2 in this more general setting. It turns out that $\mathrm{Rep}(\Pi Q, \alpha)$ is in fact too big, and we need to pass to the nilpotent variety

$$\Lambda^\alpha = \{x \in \mu_\alpha^{-1}(0) \mid x \text{ is nilpotent}\}.$$

Here nilpotent means that there exists some N such that for any path of length N, multiplying the corresponding matrices gives 0. This variety turns out to have nice geometric properties:

Theorem 6.1. *The subvariety $\Lambda^\alpha \subset \mathrm{Rep}(\overline{Q}, \alpha)$ is Lagrangian.*

The proof of the theorem is divided into two parts: first we prove that Λ^α is of pure dimension $\frac{1}{2}\dim \mathrm{Rep}(\overline{Q}, \alpha)$, and then we prove it's isotropic. The material in this section can be found in Lusztig [4].

6.1. **Counting dimensions.** The technique to prove half dimensionality is to use so called Hecke correspondences, combined with induction. The idea is to construct a diagram

$$\begin{array}{ccc} & \Lambda(\alpha, \alpha + d\epsilon_i)^* & \\ \swarrow^{p} & & \searrow^{q} \\ \Lambda^\alpha \times \Lambda^{d\epsilon_i} & & \Lambda^{\alpha + d\epsilon_i} \end{array}$$

for some variety $\Lambda(\alpha, \alpha + d\epsilon_i)^*$, and natural maps p and q. Then Λ^α can be stratified in such a way that these maps become smooth fibrations on each stratum separately. This induces a bijection between the irreducible components of the corresponding strata, allowing one to calculate the dimension of Λ^α by going up and down the diagram using a suitable induction hypothesis. The variety $\Lambda^{\alpha, \alpha + d\epsilon_i}$ consists of couples (x, W), where $x \in \Lambda^{\alpha + d\epsilon_i}$, and W is a subspace of $k^{\alpha + d\epsilon_i}$ of dimension α that is x-stable. The variety $\Lambda(\alpha, \alpha + d\epsilon_i)^*$ consists of quadruples (x, W, ρ, ψ), where x and W are as above and the maps ρ and ψ are linear isomorphisms fixing the fact that there is no natural map $\Lambda^{\alpha, \alpha + d\epsilon_i} \to \Lambda^\alpha \times \Lambda^{d\epsilon_i}$. These maps define p, and q is just projection onto the first component. It is clear that neither of these maps have to be locally trivial. The stratification of Λ^α that fixes this is given by

$$\Lambda^\alpha = \bigcup_{i \in Q_0} \Lambda^\alpha_{i, \geq 1},$$

where

$$\Lambda^\alpha_{i, \geq 1} = \bigcup_{l \geq 1} \left\{ x \mid \mathrm{codim}\ _{(k^\alpha)_i} \mathrm{Im}\Big(\bigoplus_{\substack{a \in \overline{Q} \\ h(a) = i}} x_a \Big) = l \right\}$$

From this it is rather easy to deduce that $q : N \to \Lambda^{\alpha + d\epsilon_i}_{i;d}$ is a principal $G_\alpha \times G_{d\epsilon_i}$-bundle, and $p : N \to \Lambda^\alpha_{i;0}$ is a smooth map with connected fibers of dimension $\sum_{j \in Q_0} \alpha_j^2 + d\alpha_i + d^2$. Now we use the following proposiiton.

Proposition 6.2. *If $u : Y \to X$ is a dominant flat map of finite type, such that the generic fibers are irreducible, then there is a bijection between the irreducible components of Y and those of X.*

Proof. A flat map of finite type is open, so the image of u is open and dense, and one can replace X by this image, since this does not change irreducible components. Openness also implies that if U is dense in X, so is $u^{-1}(U)$ in Y. So X can be replaced by a dense open set. By taking this to be a disjoint union of irreducible open sets, one can reduce to X irreducible. Now suppose Y contains non-empty opens V_1, V_2 with empty intersection. Then by openness, $u(V_1)$ and $u(V_2)$ contain the generic point of X, so V_1 and V_2 intersect the generic fibre of u, which is irreducible, so Y is irreducible. □

Example 6.3. Looking at the projection of $\mathbb{V}(xy) \subset \mathbb{A}^2$ onto the x-axis shows that flatness is really necessary.

Now a locally trivial fibration is flat, so the proposition gives a bijection $\kappa_{i;d}^{\alpha}$: $\mathrm{Irr}(\Lambda_{i;0}^{\alpha}) \leftrightarrow \mathrm{Irr}(\Lambda_{i;d}^{\alpha+d\epsilon_i})$. If we induct on α, we can assume Λ^{γ} is Lagrangian for all $\gamma < \alpha$. By using $\kappa_{i;d}^{\alpha-d\epsilon_i}$ the dimension of an irreducible component of Λ^{α} can easily be calculated from the knowledge that q is a principal bundle and the fiber dimension of p. This proof also automatically gives us, for any $i \in Q_0$, a canonical bijection

$$\mathrm{Irr}(\Lambda^{\alpha}) \leftrightarrow \bigsqcup_{d \geq 0} \mathrm{Irr}\Lambda_{i;d}^{\alpha}.$$

6.2. **Isotropicity.** What follows is an easier argument showing Lusztig's nilpotent variety is isotropic.

Proposition 6.4. *Let (X, ω, G, μ) be a Hamiltonian G-space, and let $Y \subset X$ be an isotropic subvariety. Then*

$$\overline{G(Y \cap \mu^{-1}(0))}$$

is also isotropic.

Corollary 6.5. *Lusztig's nilpotent variety Λ^{α} is isotropic.*

Proof. Let $X = \mathrm{Rep}(\overline{Q}, \alpha)$, $G = GL(\alpha)$, and let $W = (V = W^0 \supset W^1 \supset \ldots \supset W^m = 0)$ be a standard flag of coordinate spaces in $V = k^{\alpha}$. Notice that unlike in the case of single vector spaces (one point quivers), there can be more than one combinatorial type of complete flag. Now set

$$Y_W = \{x \in X \mid \forall i : x(W^i) \subset W^i\}.$$

Remember that the symplectic form on X is given by

$$\omega(x, y) = \sum_{a \in Q} \mathrm{Tr}(x_a y_{a^*}) - \mathrm{Tr}(x_{a^*} y_a),$$

and since Y_W is a linear (i.e. flat) subspace of X, we see that $x_a y_{a^*}$ and $x_{a^*} y_a$ are nilpotent linear transformations, so Y_W is obviously isotropic. From the proposition, we deduce that $G(Y_W \cap \mu^{-1}(0))$ is isotropic. Now the full nilpotent variety Λ^{α} is the union, over all combinatorial types of W of these $G(Y_W \cap \mu^{-1}(0))$. Elements of $G(Y_W \cap \mu^{-1}(0))$ obviously correspond to nilpotent representations, and with every nilpotent representation we can associate a standard flag (up to conjugation), by extension of what one does in linear algebra. Since there are only a finite number of combinatorial types, the union, and thus Λ^{α} is isotropic. □

Remark 6.6. Let's explain in more detail the correspondence between flags and nilpotent representations. Given a sequence of (not necessarily different) vertices $i = (i_1, \ldots, i_m)$ and a sequence $a = (a_1, \ldots, a_m)$ of strictly positive integers such that $\sum_{l:i_l=k} a_l = \alpha_k$ for all $k \in Q_0$, we define a flag of type (i, a) in some Q_0-graded vector space $V \cong k^{\alpha}$ to be a sequence $\phi = (V = V^0 \supset V^1 \supset \ldots \supset V^m = 0)$ of Q_0-graded subspaces of V, such that for any $l = 1, \ldots, m$, the graded vector space V^{l-1}/V^l is zero in degrees $\neq i_l$, and has dimension a_l in degree i_l. For an $x \in \mathrm{Rep}(\overline{Q}, \alpha)$, ϕ is said to be x-stable if $x_a(V_{t(a)}^l) \subset V_{h(a)}^l$ for all $a \in \overline{Q}$, and all $l = 1, \ldots, m$. Denoting by $\mathcal{F}_{i,a}$ the variety of all flags of type (i, a) in V. There is an obvious $GL(\alpha)$-action on $\mathcal{F}_{i,a}$, which is easily seen to be transitive (since $\mathcal{F}_{i,a}$ is isomorphic to a product over Q_0 of usual partial flag manifolds associated to the

V_i). If there exists an $x \in \operatorname{Rep}(\overline{Q}, \alpha)$ and a flag of type (i, a) that is x-stable, it is easy to see that $x_a(V^{l-1}_{t(a)}) \subset V^l_{h(a)}$ and thus x has to be nilpotent (notice this also proves that the nilpotency class of x is at most $\sum_i \alpha_i$, showing that Λ^α is an actual affine variety). Conversely, if $x \in \operatorname{Rep}(\overline{Q}, \alpha)$ is nilpotent, then there exists a flag ϕ of some type (i, a) that is x-stable (the proof is an easy induction argument, see [4]). It is also obvious that for some fixed α, the number of couples (i, a) is finite.

The proof of Proposition 6.4 itself follows easily from the following lemma.

Lemma 6.7. *Let $\pi : Z \to X$ be a morphism of smooth algebraic varieties. Assume X is symplectic with symplectic form ω. If $\pi^*\omega = 0$, then $\overline{\pi(Z)}$ is isotropic.*

Proof. Let $Y = \overline{\pi(Z)}$, and let $Y' \subset Y$ be an arbitrary smooth irreducible open subset of Y. Picking one such Y' for each irreducible component, it suffices to prove that $\omega|_{Y'} = 0$. By cutting out $Y \backslash Y'$ from X, we can assume Y is smooth and irreducible. Now we can reduce even further by using generic smoothness (see below) to assume π defines a smooth map $\pi^{-1}(Y) \to Y$. Picking a $y \in Y$ and a z such that $\pi(z) = y$, by smoothness the map $d\pi_z : T_zZ \to T_yY$ is surjective. If $\pi^*\omega_z = 0$, then $\omega_y = 0$, proving the proposition. □

Proposition 6.8 (Hartshorne [3], III Cor. 10.7)**.** *Let $f : X \to Y$ be a morphism of varieties over an algebraically closed field k of characteristic 0, and assume that X is nonsingular. Then there is a nonempty open subset $V \subset Y$ such that $f : f^{-1}V \to V$ is smooth.*

Proof. (of Proposition 6.4) Consider the action map

$$\pi : G \times (Y \cap \mu^{-1}(0)) \to X.$$

It does not immediately fulfill the conditions of the lemma, since $Y \cap \mu^{-1}(0)$ can badly fail to be smooth. Since the smooth points form an open dense subset of $Y \cap \mu^{-1}(0)$ however, this is not a problem and we can use the lemma. Given such a point x, $\xi_{1,2} \in T_eG, v_{1,2} \in T_x(Y \cap \mu^{-1}(0))$, it is enough to check that

$$\pi^*\omega_{(e,x)}((\xi_1, v_1), (\xi_2, v_2)) = \omega_x(\xi_{1,x} + v_1, \xi_{2,x} + v_2) = 0,$$

since ω is G-invariant. This follows from the computation

$$\begin{aligned} \omega_x(\xi_{1,x} + v_1, \xi_{2,x} + v_2) &= \omega_x(\xi_{1,x}, \xi_{2,x} + v_2) + \omega_x(v_1, \xi_{2,x}) + \omega_x(v_1, v_2) \\ &= \omega_x(\xi_{1,x}, \xi_{2,x} + v_2) + \omega_x(v_1, \xi_{2,x}), \end{aligned} \tag{6.1}$$

where the third term in the RHS is zero because Y is isotropic. Now $v_{1,2}$ are tangent to $\mu^{-1}(0)$, as are $\xi_{1,2}$, by G-equivariance, so they are all contained in the kernel of $d\mu_x$. Remembering the moment map condition

$$\omega_x(v, \xi_x) = \langle d\mu_x(v), \xi \rangle,$$

this immediately implies the desired result. □

6.3. **Link with quantum groups.** The results obtained in the previous section allow one to obtain analogues of the natural bijections we had in the Dynkin case for any quiver without loops Q, see [4]. We will only sketch these connections. To Q one can associate a generalized (symmetric) Cartan matrix in the same way as before, though off diagonal elements a_{ij} are now given by minus the number of (unoriented) edges between i and j, which in turn gives rise to the associated Kac-Moody Lie algebra $\mathfrak{g}$. Using the triangular decomposition of $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, Lusztig was able (as in the Dynkin setting) to construct a canonical basis of $\mathcal{U}_q(\mathfrak{n}_+)$, using an isomorphism of the quantum group with a subalgebra of the generalized Hall algebra (an object similar to the one we saw in the Dynkin case). By a suitable categorification, one is led to the Hall category, which is a category of constructible sheaves on the moduli stack $\mathrm{Rep}(Q, \alpha)/GL(\alpha)$. The simple perverse sheaves therein then give rise to the canonical basis $\mathcal{B}$.

The bijection between $\mathcal{B}_\alpha$ and $\mathrm{Irr}(\Lambda^\alpha)$ can be seen to arise from Kashiwara's theory of crystals. A $\mathfrak{g}$-crystal is a set $\mathbb{B}$ equipped with a number of maps: four families indexed by the vertices of Q, and one weight map. These crystals serve as combinatorial skeletons of $\mathfrak{g}$-modules: the set $\mathbb{B}$ 'corresponds' to a basis, the families of maps to the action of the Chevalley generators on $\mathbb{B}$ and the weight map gives the weights of the basis elements. Given a highest weight integrable $\mathfrak{g}$-module M, Kashiwara also introduced the notion of crystal basis which consists of a pair $(\mathcal{L}, \mathbb{B})$, that is in some appropriate sense a basis of M in the $q \to 0$ limit. With such a basis one can immediately construct a $\mathfrak{g}$-crystal by endowing $\mathbb{B}$ with some obvious maps. We are interested in crystal bases associated to $\mathcal{U}_q(\mathfrak{n}_+)$ itself, which can be proven to be unique up to suitable isomorphism. To the canonical basis $\mathcal{B}$ of $\mathcal{U}_q(\mathfrak{n}_+)$ one can associate a crystal basis, which in turn gives rise to a crystal, denoted $\mathbb{B}^+(\infty)$. By uniqueness, there is thus a bijection $\mathcal{B} \leftrightarrow \mathbb{B}^+(\infty)$. Kashiwara and Saito proved a theorem giving sufficient conditions for a crystal to be isomorphic to $\mathbb{B}^+(\infty)$.

On $\mathrm{Irr}(\Lambda) = \bigsqcup_\alpha \mathrm{Irr}(\Lambda^\alpha) = \bigsqcup_\alpha \bigsqcup_{d \geq 0} \mathrm{Irr}(\Lambda^\alpha_{i;l})$, one can build natural maps using the bijections $\kappa^\alpha_{i;d}$, which endow $\mathrm{Irr}(\Lambda^\alpha)$ with the structure of a $\mathfrak{g}$-crystal. Using his theorem, Kashiwara then succeeded in proving that this crystal comes from the crystal basis associated to $\mathcal{B}$, so is in fact isomorphic to $\mathbb{B}^+(\infty)$. By combining these results we thus obtain beautiful algebro-geometric bijections, generalizing the ones we had before in the setting of Dynkin quivers

$$\mathcal{B} \leftrightarrow \mathbb{B}^+(\infty) \leftrightarrow \mathrm{Irr}(\Lambda).$$

Details for this last bit, including connections to Ringel-Hall theory and perverse sheaves, can be found in Schiffmann [7, 8].

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