ON THE GLOBAL RIGIDITY OF PROJECTIVE SPACE

THEO RAEDSCHELDERS

Abstract. Using the characterisation of projective space by Kobayashi-Ochiai, we give an elementary proof of the global rigidity of projective space in the algebro-geometric setting.

Throughout we work over the complex numbers. Projective space $\mathbb{P}^n$ is locally rigid, meaning that it has no non-trivial infinitesimal deformations. Indeed, classical deformation theory tells us that these deformations are parametrised by the first sheaf cohomology group of the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$, and a computation using the Euler sequence shows that $\mathrm{H}^1(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) = 0$.

On the other hand, it is perfectly possible for locally rigid varieties to admit global deformations, the following being a classical example: set

$$\mathcal{X} = \{(y, x, t) \in \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{A}^1 \mid y_0^2 x_0 - y_1^2 x_1 - t y_0 y_1 x_2 = 0\},$$

and consider the morphism $\pi : \mathcal{X} \to \mathbb{A}^1 : (y, x, t) \to t$. It is then easy to check that for $t \neq 0$, the fibre $\mathcal{X}_t \cong \mathbb{P}^1 \times \mathbb{P}^1$, but $\mathcal{X}_0 \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$, the second Hirzebruch surface. Since $\mathrm{H}^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1}) = 0$ just like for projective space, this shows that $\mathbb{P}^1 \times \mathbb{P}^1$ is locally but not globally rigid.

This behaviour does not occur for projective space however, according to the following theorem which is due to Siu.

Theorem 0.1. [3] *Let $\pi : \mathcal{X} \to \Delta$ be a holomorphic family of compact complex manifolds where $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, such that $\mathcal{X}_t$ is biholomorphic to $\mathbb{P}^n$, for $t \neq 0$. Then $\mathcal{X}_0$ is biholomorphic to $\mathbb{P}^n$.*

In particular, Siu does not assume from the outset that the special fibre $\mathcal{X}_0$ is Kähler. In this note we want to show that if one stays inside the realm of complex projective varieties, there is an elementary proof, based on the following characterisation of projective space due to Kobayashi and Ochiai.

Theorem 0.2. [2] *Let X denote a smooth complex variety of dimension n, with an ample line bundle $\mathcal{L}$. If*

$$c_1(X) \geq (n + 1)c_1(\mathcal{L}),$$

then X is isomorphic to projective space $\mathbb{P}^n$.

Using this characterisation, we can now prove the following theorem. Note in particular that in contrast to Theorem 0.1 we assume from the start that we are dealing with a smooth proper family. This proof is analogous to [4, Proposition 1.3].

Theorem 0.3. *Let $\pi : \mathcal{X} \to \operatorname{Spec}(C)$ denote a smooth proper morphism of schemes over a curve C such that the generic fibre is isomorphic to $\mathbb{P}^n$. Then the special fibre is also isomorphic to $\mathbb{P}^n$.*

Proof. Since the problem is local on the target, we work over $\operatorname{Spec}(R)$, where R is a discrete valuation ring with fraction field K and residue field $\mathbb{C}$. Accordingly we denote the generic fibre of π by $\mathcal{X}_K$ and the special fibre by X.

There is an exact sequence of Picard groups

$$\operatorname{Pic}(R) \to \operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(\mathcal{X}_K) \to 0,$$

and since R is a DVR, $\operatorname{Pic}(R) = 0$. Hence we obtain a specialization morphism

$$\operatorname{sp} : \operatorname{Pic}(X_K) \to \operatorname{Pic}(X).$$

Since $X_K \cong \mathbb{P}^n$, we can look at the line bundles $\mathcal{L}_i = \operatorname{sp}(\mathcal{O}(i))$ on X for $i = 0, \dots, n$. Euler characteristics are constant in families [1, Ch.2, §3.8], and so we find that

$$\chi(\mathcal{L}_i) = 1, \text{ for all } i, \qquad (1)$$
$$\chi(\mathcal{L}_j, \mathcal{L}_i) = 0, \text{ for } j > i.$$

Remark that the $\mathcal{L}_i$ are non-isomorphic, $\mathcal{L}_0 = \mathcal{O}_X$ and $\mathcal{L}_1$ is ample.

If $\mathcal{L}_1 = \mathcal{O}_X(H)$, then $\mathcal{L}_i = \mathcal{O}_X(iH)$, and for an arbitrary $\mathcal{O}_X(aH)$ we find using Riemann-Roch that

$$\chi(\mathcal{O}_X(aH)) = \frac{\deg(H^n)}{n!}a^n + \frac{\deg(H^{n-1} \cdot c_1(X))}{2(n-1)!}a^{n-1} + \cdots + \chi(\mathcal{O}_X). \qquad (2)$$

Consider $P(x) = \chi(\mathcal{O}_X(xH))$ as a polynomial in x, which is hence of degree n. Then by (1), the numbers $-1, \dots, -n$ are roots of this polynomial, and they are hence all the roots. In other words,

$$P(x) = c\prod_{i=1}^{n}(x+i) = cx^n + \frac{cn(n+1)}{2}x^{n-1} + \cdots + cn!,$$

for some constant c. Since $\chi(\mathcal{O}_X) = 1$ (again by (1)), we find that $c = 1/n!$. Hence, from (2), we conclude that

$$\deg(H^n) = 1$$
$$\deg(H^{n-1} \cdot c_1(X)) = n + 1.$$

Hence $c_1(X) = (n+1)c_1(H)$ and so we can apply Theorem 0.2 with $\mathcal{L} = \mathcal{O}_X(H)$ to conclude. □

Using some standard base change arguments, one can obtain the same result over an arbitrary field of characteristic 0.

References

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