RATIONALITY OF COUPLES OF 3×3 MATRICES UNDER SIMULTANEOUS CONJUGATION

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1. Introduction

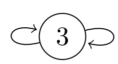
The aim of this note is to give an elementary proof of the following theorem, established by Formanek in 1979 [2]. Throughout, we work over the field of complex numbers.

Theorem 1.1. *The moduli space* $\mathcal{K}_3 = \mathtt{M}_3 \times \mathtt{M}_3/\mathtt{PGL}_3$ *of couples of* 3×3 *matrices under simultaneous conjugation is a rational variety.*

For general n, the rationality of $\mathcal{K}_n = \mathtt{M}_n \times \mathtt{M}_n/\mathtt{PGL}_n$ is still a wide open problem. In [3], Formanek succeeded in proving the $n = 4$ case, and for $n = 5, 7$, stable rationality was shown by Bessenrodt and Le Bruyn in [1]. Using a result by Schofield [5], this establishes stable rationality for all n dividing $420 = 2^2 \cdot 3 \cdot 5 \cdot 7$.

2. The proof

We will construct a rational map from $\mathbb{A}^{12}$ to $\mathcal{K}_3$. This map factors through a quotient of $\mathbb{A}^{12}$ by the group generated by $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ and S_3 embedded in $\mathtt{GL}_3$. The action of the diagonal matrices will of course be trivial. Denoting by Q the two-loop quiver



with dimension vector $\alpha = (3)$, it thus suffices to show that $\mathtt{iss}_\alpha Q = \mathtt{rep}_\alpha Q/PGL_\alpha$ is rational. Define a morphism $\pi : \mathbb{A}^{12} \to \mathtt{iss}_\alpha Q$ by associating to the first 3 coordinates the corresponding diagonal matrix, and to the other 9 a general matrix. Suppose (A, B) is a representation of Q. The condition that A has distinct eigenvalues is an open condition, so we work on the open subset $U \subset \mathtt{rep}_\alpha Q$ for which each point has a representative in its orbit of the form $(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3), X)$ with the λ_i all distinct. The morphism π thus induces a surjective morphism from V to $\overline{U}$, where V denotes the open subset of $\mathbb{A}^{12}$ with first 3 coordinates distinct, and $\overline{U}$ is the open image of U under the corresponding quotient map.

Theorem 2.1. *The morphism* $\mathbb{A}^{12} \to \mathtt{iss}_\alpha Q$ *induces a morphism of a GIT-quotient of* $\mathbb{A}^{12}$ *by the subgroup* $G \subset \mathtt{GL}_3$ *generated by the diagonal matrices and* S_3 *embedded in* $\mathtt{GL}_3$*. The induced morphism* $\mathbb{A}^{12}/G \to \mathtt{iss}_\alpha Q$ *is a birational map.*

Proof. The coordinate functions on $\mathbb{A}^{12}$ will be given by $(\lambda_k, x_{i,j})$ with $1 \leq i, j, k \leq 3$. So suppose that 2 points p and p' of $\mathbb{A}^{12}$ are mapped to the same point. We then have for some $g \in \mathtt{GL}_3$ that $g\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)g^{-1} = \mathrm{diag}(\lambda'_1, \lambda'_2, \lambda'_3)$. Eigenvalues are determined up to a permutation, which means that there is a $\sigma \in S_3$ such that $\sigma\,\mathrm{diag}(\lambda'_1, \lambda'_2, \lambda'_3)\sigma^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. From this it follows that

$g\sigma \in \mathtt{Stab}(\mathrm{diag}(\lambda'_1, \lambda'_2, \lambda'_3))$, which is equal to T_3, the maximal torus in $\mathtt{GL}_3$. Because we are looking at the action of $\mathtt{PGL}_3$ by simultaneous conjugation, it follows that the induced map from $\overline{V} \subset \mathbb{A}^{12}/G$ to $\overline{U}$ is an isomorphism, so the morphism $\mathbb{A}^{12}/G \dashrightarrow \mathtt{iss}_\alpha Q$ is in fact a birational map. □

This means that, in order to prove rationality of $\mathcal{K}_3$, it is sufficient to determine the fraction field of $\mathbb{C}[\mathbb{A}^{12}/G]$. It is easier to decompose $\mathbb{A}^{12}$ into smaller representations of G:

- The subspace $V_1 = (\lambda_1, \lambda_2, \lambda_3, 0)$, on which the action of G is just the action of S_3.
- The subspace generated by $V_2 = x_{1,1}, x_{2,2}, x_{3,3}$, on which the action of G is also just the action of S_3.
- The subspace generated by $V_3 = x_{i,j},\ i \neq j, 1 \leq i, j \leq 3$.

The third case is the only non-trivial one. In the other cases, $\mathbb{C}[V_i]$ is just the polynomial ring in 3 variables. For V_3, it is easier to calculate $\mathbb{C}[V_3]^{T_3}$ and to find the S_3-invariant functions of this subring of $\mathbb{C}[V_3]$.

Theorem 2.2. *The ring $\mathbb{C}[V_3]^{T_3}$ is generated by the following 5 functions*

- $u_1 = x_{1,2}x_{2,1}$
- $u_2 = x_{2,3}x_{3,2}$
- $u_3 = x_{1,3}x_{3,1}$
- $v_1 = x_{1,2}x_{2,3}x_{3,1}$
- $v_2 = x_{1,3}x_{3,2}x_{2,1}$

This ring satisfies one relation, given by $u_1u_2u_3 - v_1v_2$.

Proof. It is clear that the 5 functions found here are indeed invariant under the action of G. Suppose now that $f \in \mathbb{C}[V_3]^{T_3}$. Because the eigenvectors of this action are the monomials, we may assume that f is a monomial. We may divide f by the u_i until we get a monomial with only 3 variables. Suppose now for example that f contains $x_{1,2}^k$ with k maximal. Because we have divided by the u_i, the only way that f can be invariant is if $x_{2,3}^k$ is also in f. But then, by the same reasoning, $x_{3,1}^k$ must also belong to f and therefore $f = (x_{1,2}x_{2,3}x_{3,1})^k$. For $x_{2,1}$ it is similar, so f belongs to $\mathbb{C}[u_1, u_2, u_3, v_1, v_2]$. It is furthermore clear that the generators of this ring satisfy $u_1u_2u_3 - v_1v_2$. The Krull dimension of $\mathbb{C}[V_3]^{T_3}$ has to be 4 (the scalar matrices work trivially), so this is indeed the only relation. □

In order to find the S_3-invariant functions of $\mathbb{C}[V_3]^{T_3}$, it suffices to find the S_3-invariant functions of $\mathbb{C}[u_1, u_2, u_3]$ and $\mathbb{C}[v_1, v_2]$ separately, for the relation $u_1u_2u_3 - v_1v_2$ is easily seen to be S_3-invariant.

Lemma 2.3. *As S_3-representations, $\oplus_{i=1}^3 \mathbb{C}u_i$ is isomorphic to the permutation representation and $\oplus_{i=1}^2 \mathbb{C}v_i$ is the sum of the trivial representation and the sign representation.*

Proof. The character of $\oplus_{i=1}^3 \mathbb{C}u_i$ is $(3, 1, 0)$ and the character of $\oplus_{i=1}^2 \mathbb{C}v_i$ is $(2, 0, 2)$. The claim follows. □

From this it follows that $\mathbb{C}[V_3]^G$ is a polynomial ring in 4 variables, given by

$$u_1 + u_2 + u_3 \qquad u_1u_2 + u_2u_3 + u_3u_1 \qquad u_1u_2u_3 = v_1v_2 \qquad v_1 + v_2$$

Summarizing, we have

Theorem 2.4. *The ring $\mathbb{C}[V]^G$ is a polynomial ring in 10 variables.*

In order to establish rationality of $\mathtt{iss}_\alpha Q$, one still needs to prove that $\mathbb{C}(\mathbb{A}^{12})^G = \mathbb{C}(\mathbb{A}^{12}/G)$. To accomplish this, we use Proposition 6.2 from [4].

Theorem 2.5. [4] *Let G be a linearly reductive group acting on an affine variety X and suppose that there exists a stable point for this action. Then we have that $\mathbb{C}(X)^G = Frac(\mathbb{C}[X]^G)$.*

Corollary 2.6. *$\mathbb{C}(V)^G$ is purely transcendental of degree 10.*

Proof. We need to find a stable point of $\mathbb{A}^{12}$. Take $\lambda_i = 0, x_{i,i} = 0, x_{i,j} = 1$. It is clear that S_3 works trivially on this element. Elements of T_3 send

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & t_1t_2^{-1} & t_1t_3^{-1} \\ t_2t_1^{-1} & 0 & t_2t_3^{-1} \\ t_3t_1^{-1} & t_3t_2^{-1} & 0 \end{bmatrix},$$

from which it follows that an element $t \in T_3$ stabilizes X if and only if $t_1 = t_2 = t_3$, which is fine because the action we are working with is not really the action of G, but the action of $G/\mathbb{C}^* I_3$. So the stabilizer of this element is indeed finite. The orbit of this point is closed, since it is determined by the equations

$$\begin{cases} x_{1,2}x_{2,1} = 1 \\ x_{1,3}x_{3,1} = 1 \\ x_{2,3}x_{3,2} = 1 \\ x_{1,2}x_{2,3} = x_{1,3}. \end{cases}$$

To see this, take a matrix that fulfills these conditions, then one finds

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & ab \end{bmatrix} \begin{bmatrix} 0 & a & ab \\ a^{-1} & 0 & b \\ a^{-1}b^{-1} & b^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a^{-1}b^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

□

References

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